Nov. 1. K-theory for symmetric monoidal category and "Plus = Q" theorem

Properties Let A, B be two Morita equivalent rings, then
$K_i(A) \cong K_i(B)$ $\forall i \geq 0$.

Recall, two rings are morita equivalent if $\exists$ a projective A-module P such that $B = \mathrm{End}_A(P)$, then P is a (B, A)-bimodule. Then $\underline{\mathrm{Mod}}(A) \simeq \underline{\mathrm{Mod}}(B)$ by

$$\underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B)$$
$$M \mapsto M \otimes_A P^*$$
$$\underline{\mathrm{Mod}}(A) \leftarrow \mathrm{Mod}(B)$$
$$N \otimes_B P \leftarrow N$$

Hence $\mathbb{P}(A) \simeq \mathbb{P}(B)$ as well, therefore the K-space $K(A)$ and $K(B)$ are homotopic equivalent.

Some constructions

1) $QC^{op} \simeq (QC)^{op}$ so $B(QC^{op}) = B(QC)^{op} \simeq BQC$
$\Rightarrow$ $K_i(C^{op}) \cong K_i(C)$ $\forall i \geq 0$

2) For two exact categories C, D. $C \oplus D$ is naturally an exact category and $Q(C \oplus D) = QC \times QD$
$\Rightarrow$ $BQ(C \oplus D) = BQC \times BQD$.
$\Rightarrow$ $K_i(C \oplus D) \cong K_i C \oplus K_i D$ $i \geq 0$.

Apply this to $\mathbb{P}(A), \mathbb{P}(B) \Rightarrow K_i(A \times B) \simeq K_i(A) \oplus K_i(B)$.

THM (Quillen - Gersten) C exact, $B \subset C$ subexact category: closed under extension and cofinal. Then BQB is homotopy equiv. to a covering space of BQC, $K_0(B) \cong \pi_1(BQB) \subseteq K_0(C) \cong \pi_1(BQC)$. and $K_i(B) = K_i(C)$ $\forall i \geq 1$.

THM (Quillen). $\forall$ associative unital ring A, $\exists$ homotopy equivalence $\Omega BQ[\mathbb{P}(A)] \simeq K_0(A) \times BGL(A)^+$, therefore
$$K_i(A) \simeq K_i(\mathbb{P}(A)) \quad \forall i \geq 0.$$

Step 1. Let $\mathcal{C}$ be a split exact category, $\mathcal{S} = \mathrm{Iso}(\mathcal{C})$.
Then $K(\mathcal{C}) = \Omega BQ\mathcal{C} \simeq B(S^{-1}S)$.
where $B(S^{-1}S)$ is a "group completion" of the H-space BS.

Step 2. $\mathcal{C} = \mathbb{P}(A)$, then $B(S^{-1}S) \simeq K_0(A) \times BGL(A)^+$

H-spaces.

Recall, an H-space (X, μ, e) is a topological space X with a multiplication map $\mu: X \times X \to X$ and a distinguished point $e \in X$ s.t. $\mu(x,e) = \mu(e,x) = x \ \forall x \in X$.

X is homotopy associative if

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu\times id"] \arrow[d, "\mu\times id"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

commutes up to homotopy.

X is homotopy commutative if $\mu \circ \tau \sim \tau$ where $\tau: X \times X \to X \times X$ with $\tau(x,y) = (y,x)$.

Ex. Topological groups ~~is~~ are H-spaces.

Ex. $\mathcal{S}$: symmetric monoidal, $B\mathcal{S}$ is a homotopy associative and commutative H-space.

If X is homotopy commutative and associative, then $\pi_0(X)$ is naturally an abelian monoid. $H_0(X,\mathbb{Z}) = \mathbb{Z}[\pi_0(X)]$ and $H_\bullet(X,\mathbb{Z}) = \bigoplus_{i\geq 0} H_i(X,\mathbb{Z})$ is an associative commutative grade $\mathbb{Z}[\pi_0(X)]$-algebra.

<u>Def:</u> An associative H-space is called group-like if $\exists$ a map $i: X \to X$ s.t.

$$X \xrightarrow{i\times id} X \times X \xrightarrow{\mu} X \quad \text{are homotopic to } id_X.$$
$$X \xrightarrow{id\times i} X \times X \xrightarrow{\mu} X$$

Note if X is group-like, then $\pi_0(X)$ is a group naturally.

<u>THM</u>. If an H-space is CW, then ~~B~~ $\pi_0(X)$ is a group $\Leftrightarrow$ X is group like.

<u>Ex.</u> $\mathcal{S} = \mathrm{Iso}(\mathcal{S})$ a symmetric monoidal ~~gro~~groupoid, then $B\mathcal{S}$ is group like $\Leftrightarrow$ $\pi_0(\mathcal{S})$ is a group.

<u>Def:</u> Let X be a homotopy associative and commutative H-space. A group completion of X is a morphism $f: X \to \hat{X}$ s.t.

1) The induces map $f_*: \pi_0(X) \to \pi_0(\hat{X})$ is a group completion of $\pi_0(X)$.
2) $\forall$ commutative ring k, the map of graded $\mathbb{Z}[\pi_0(X)]$-algebras ~~[illegible]~~ $f_\bullet: H_\bullet(X,k) \to H_\bullet(\hat{X},k)$ induces an isomorphism.

$$\begin{tikzcd} H_\bullet(X,k) \arrow[r, "f_\bullet"] \arrow[d] & H_\bullet(\hat{H},k) \\ H_\bullet(X,k)[\pi_0(X)^{-1}] \arrow[ur, dashed, "\simeq"'] & \end{tikzcd}$$

Lemma Any group-like H-space X is its own group completion. Any group completion of a group completion $f: X \to \hat{X}$ is a homotopy equivalence.

Actions on categories

Let $(S, \oplus)$ be a monoidal category. We say $(S, \oplus)$ acts on a category X by a functor $\oplus : S \times X \to X$ if $\forall\ s, t \in Ob(S),\ x \in Ob(X)$

$$s \oplus (t \oplus x) \simeq (s \oplus t) \oplus x$$

$$e \oplus x \simeq x.$$

satisfying certain coherence conditions.

Note. S act on itself by its monoidal structure $\oplus: S \times S \to S$.

Ex. X discrete, then S acts on $X \iff \pi_0(S)$ acts on $Ob(S)$.

Def: S act on X as before. Define a new category $S \ltimes X$ with $ob(S \ltimes X) = Ob(X)$, and $Hom_{S \ltimes X}(x, y) = \{(s, \varphi) \mid s \in ob(S),$
$\varphi \in Mor_X(s \oplus x, y)\}/\sim$

where $\sim$ is the equivalence relation that $(s, \varphi) \sim (s', \varphi')$ if $\exists \alpha: s \xrightarrow{\sim} s'$ an isomorphism s.t.

$$\begin{tikzcd} s \oplus x \arrow[rr, "\alpha \oplus id_x"] \arrow[dr, "\varphi"'] & & s' \oplus x \arrow[dl, "\varphi'"] \\ & y & \end{tikzcd}$$ commutes.

Def. We define $S^{-1}X := S \ltimes (S \times X)$.

If S is symmetric monoidal, we could define an action of S on $S^{-1}X$ by $s \oplus (t, x) = (s \oplus t, x)$.

Def. S acts on X homotopy invertibly if every translation functor $s\oplus : x \mapsto s\oplus x$ is a homotopy equivalence.

Ex. S acts on $X \Rightarrow S$ acts on $S^{-1}X$ homotopy invertibly.
symmetric monoidal.

If S is symmetric monoidal, then $H_0(S) = H_0(BS, \mathbb{Z}) = \mathbb{Z}[\pi_0(S)]$
S acts on $X \Rightarrow H_0(S)$ acts on $H_*(X) = H_*(BX, \mathbb{Z})$ and so acts invertibly on $H_*(S^{-1}X)$. Hence the functor $X \to S^{-1}X$
$x \mapsto (e, x)$ induces a map:

$$H_i(X)[\pi_0(S)^{-1}] \xrightarrow{(*)} H_i(S^{-1}X) \quad \forall i \geq 0.$$

THM (Quillen) Let S be symmetric monoidal with $S = \mathrm{Iso}(S)$, all translations in S are faithful. Then $(*)$ is an isomorphism.
In particular, $B(S^{-1}S)$ is a group completion of H-space BS.

Note: all translations in S are faithful means $\forall s \in Ob(S)$,
$$s\oplus : \mathrm{Aut}_S(t) \to \mathrm{Aut}_S(s\oplus t) \text{ is injective.}$$
$$u \mapsto \mathrm{id}_s \oplus u$$

Rmk. 1) There are two distinguished type of morphisms in $S^{-1}S$.
Type I $(f_1, f_2) : (m_1, m_2) \to (n_1, n_2)$ arising from the inclusion $S \times S \hookrightarrow S^{-1}S$.
Type II : $s\oplus : (m, n) \mapsto (s\oplus m, s\oplus n)$.
If transitions in S is faithful, then any morphism in $S^{-1}S$ defines a triple (s, f, g) uniquely up to unique isomorphism.
2) $S^{-1}S$ is symmetric monoidal with
$$(m_1, m_2) \oplus (n_1, n_2) = (m_1 \oplus n_1, m_2 \oplus n_2).$$

and the natural inclusion $S \hookrightarrow S^{-1}S$: $m \mapsto (m, e)$

is a monoidal functor.

3) The induced map $BS \to BS^{-1}S$ is a morphism of H-spaces.

<u>K-groups of a symmetric monoidal groupoid.</u>

Let $(S, \oplus)$ be a symmetric monoidal groupoid, define

$K_i^{\oplus}(S) := \pi_i[B(S^{-1}S)]$

Back to "plus construction"

Let $\mathbb{F}(A) \subset \mathbb{P}(A)$ be the full subcategory of free modules

Assume A satisfies invariant basis property.

Consider the associated groupoid. $\mathrm{Iso}(\mathbb{F}(A))$

<u>Note</u> $\mathrm{Iso}(\mathbb{F}(A)) \simeq S$ has objects $Ob(S) = \{0, A, A^2, \cdots\}$

$$\mathrm{Hom}_S(A^n, A^m) = \begin{cases} \emptyset & n \neq m \\ GL_n(A) & n = m. \end{cases}$$

$\therefore S = \coprod_{n \geq 0} GL_n(A)$

and $BS = \coprod_{n \geq 0} BGL_n(A)$. <u>Note</u>: S has a monoidal structure

$A^n \oplus A^m = A^{m+n}$ & $\alpha \oplus \beta = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$.

<u>THM</u> (Quillen) If $S = \coprod_{n \geq 0} BGL_n(A)$, then $K^{\oplus}(S) = B(S^{-1}S)$

is a group completion of $BS \simeq \coprod_{n \geq 0} BGL_n(A)$, in fact

$$B(S^{-1}S) \simeq \mathbb{Z} \times BGL(A)^+$$

<u>Pf:</u> We want to use the previous theorem as well as Quillen's recognition criterion: the map $BGL(A) \to BGL(A)^+$ is universal for all maps from $BGL(A)$ to H-spaces.

Fact: Consider a map $f: BGL(A) \to H$, suppose $f_*: H_\cdot(BGL(A), \mathbb{Z}) \xrightarrow{\sim} H_\cdot(H, \mathbb{Z})$ is an isomorphism, then f is acyclic and the induced map

$$\bar{f}: BGL(A)^+ \xrightarrow{\simeq} H$$ is a homotopy equivalence.

Pf: Follows from if X, Y are H spaces with homotopy type of CW complexes, then any map $f: X \to Y$ induces isomorphisms on homology ~~induces~~ is a homotopy equivalence.

Now, $\forall n \geq 1$, we have a natural group homomorphism.

$$\eta_n : GL_n(A) = Aut_S(A^n) \longrightarrow Aut_{S^{-1}S}(A^n, A^n).$$
$$g \longmapsto (g, 1).$$

Hence
$$\begin{tikzcd} GL_n(A) \arrow[r, "\eta_n"] \arrow[d, "A\oplus"'] & Aut_{S^{-1}S}(A^n, A^n) \arrow[d, "(A,A)\oplus"] \\ GL_{n+1}(A) \arrow[r, "\eta_{n+1}"] & Aut_{S^{-1}S}(A^{n+1}, A^{n+1}) \end{tikzcd}$$

where $A\oplus$ and $(A,A)\oplus$ are translation maps.

From the commutativity of this diagram. we get a sequence of functors
$$\begin{tikzcd} GL_n(A) \arrow[d, "A\oplus"'] \arrow[dr, "\eta_n"] & \\ GL_{n+1}(A) \arrow[r, "\eta_{n+1}"] \arrow[d, "A\oplus"'] & S^{-1}S \\ GL_{n+2}(A) \arrow[ur, "\eta_{n+2}"'] & \end{tikzcd}$$

Hence natural transformations ~~induces~~ $\eta_n \Rightarrow \eta_{n+1} \circ (A\oplus)$ induces a homotopy diagram of spaces

$$\begin{tikzcd} BGL_n(A) \arrow[d, "A\oplus"'] \arrow[dr, "B\eta_n"] & \\ BGL_{n+1}(A) \arrow[r, "B\eta_{n+1}"] \arrow[d] & B(S^{-1}S) \\ {} & \end{tikzcd}$$

Hence we get a map $BGL(A) = \operatorname{colim} BGL_n(A) \xrightarrow{\eta} B(S^{-1}S)$ where η lands in the identity component Y_S of $B(S^{-1}S)$, which is also an H-space. Hence it suffices to show $H_*(BGL(A)) \xrightarrow{\simeq} H_*(Y_S)$.

Let $a = [A] \in \pi_0(BS) = \pi_0 S$ denote the class of A in $Ob(S)$

Note we have $H_*(B(S^{-1}S)) \simeq H_*(BS)[\pi_0(S)^{-1}]$

and $\pi_0(S) = \{a^n\}_{n\geq 0}$ is generated by a.

The localization above is given by the colimit of the map $H_*(BS) \to H_*(BS)$ from the translation $A\oplus : S \to S$,

i.e. $H_*(BS)[\pi_0(S)^{-1}] = \operatorname{colim}\{H_*(BS) \xrightarrow{a} H_*(BS) \xrightarrow{a} \cdots\}$

Hence $H_*(B(S^{-1}S)) \simeq H_*(Y_S) \otimes \mathbb{Z}[a, a^{-1}]$

Therefore, $H_*(Y_S) \simeq \operatorname{colim} H_*(BGL_n(A)) = H_*(BGL(A))$ □

<u>Rmk</u>. Consider $S = \mathrm{Iso}(\underline{Fin}) \simeq \coprod_{n\geq 0} S_n$ (groupoid of finite sets), we have the transition function $\{1\} : S \to S$ which yields the inclusion $S_n \hookrightarrow S_{n+1}$. Assemble these maps gives $BS_\infty \to B(S^{-1}S)$

Hence $B(S^{-1}S) = K^{\oplus}(\mathrm{Iso}(\underline{Fin})) \simeq \mathbb{Z} \times BS_\infty^{+}$.

<u>THM</u> (Barratt - Priddy - Quillen - Segal). $\exists$ isomorphism $K_i(\mathrm{Iso}(\underline{Fin})) \simeq \pi_i^{st}(\mathbb{S})$, where $\pi_i^{st}(\mathbb{S}) = \varinjlim_{n\geq 0} \pi_{i+n}(\mathbb{S}^n)$ are the stable homotopy groups of spheres.

Hence $\Omega^\infty S^\infty = \lim_{n\to\infty} \Omega^n S^n \simeq K(\mathrm{Iso}(\underline{\mathrm{Fin}}))$.

Proving "plus = Q" theorem.

We just need to show the second step, i.e "$Q = S^{-1}S$"

Thm (Quillen) Let $\mathcal{C}$ be a split exact category, let $S = \mathrm{Iso}(\mathcal{C})$. Then $\Omega BQ\mathcal{C} \simeq B(S^{-1}S)$. Hence $K_i(\mathcal{C}) \simeq K_i^{\oplus}(\mathcal{C})$ $\forall i \geq 0$.

idea: We will construct a homotopy fibration sequence

$$B(S^{-1}S) \to E \to BQ\mathcal{C}$$

with E being contractible. In fact $E = B(S^{-1}X)$ for certain category $X = Ex(\mathcal{C})$, the category of exact diagrams in $\mathcal{C}$.